# Practice 1.2: Simplex method
## *Combinatorial optimization*

G.Guérard

### Complementary slackness theorem

1. Begin with a "guess" for Primal.
2. Check feasibility. If not, stop. It cannot be a solution.
3. If so, note positive variables (implies binding dual constraints) and constraints with slack (implies zero dual values).
4. Use non-zero dual variables to solve the equations in dual. (Typically equal number of equations and unknowns.)
5. Check feasibility of this solution (are variables nonnegative? Are the dual constraints corresponding to zero primal variables satisfied?) If yes, then you have solutions to both Primal and Dual. If not, original guess is not a solution.

### First example

One of the major theorems in the theory of duality in Linear Programming is the **Complementary Slackness Theorem**. This theorem allows us to find the optimal solution of the dual problem when we know the optimal solution of the primal problem (and vice versa) by solving a system of equations formed by the decision variables (primal and dual) and constraints (primal and dual model).

The importance of this theorem is that it facilitates the resolution of the models of linear optimization, allowing you to find the simplest model to address (from the algorithmic point of view) because either way you will get the results of the associated equivalence model (may it be a primal or dual model).

Let us consider the following Linear Programming model (here in after primal) in 2 variables whose optimal solution is **X=14/5** and **Y=8/5** with optimal value **V(P)=20.8**.

$$\begin{array}{ll} \text{MAX} & 4X + 6Y \\ \text{S.A.} & 2X + 4Y \leq 12 \\ & 4X + 3Y \leq 16 \\ & X \geq 0 \quad Y \geq 0 \end{array}$$

The dual model associated with the primal model is:

$$\begin{array}{ll} \text{Min} & 12A + 16B \\ \text{s.a.} & 2A + 4B \geq 4 \\ & 4A + 3B \geq 6 \\ & A, B \geq 0 \end{array}$$

Then the Complementary Slackness Theorem shows us the following relationships:

$(2X + 4Y - 12)A = 0$
$(4X + 3Y - 16)B = 0$
$(2A + 4B - 4)X = 0$
$(4A + 3B - 6)Y = 0$

As we know **X=14/5** and **Y=8/5** (primal optimal solution). If we replace these values of X and Y in the third and fourth equation we generate a 2×2 system of equations in terms of A and B whose solution corresponds to **A=6/5** and **B=2/5** (a feasible and optimal solution of the dual model). If we subsequently evaluate the objective function in the dual problem of this solution we obtain: **V(D)=12(6/5)+16(2/5)=20.8** which is similar to the primal problem's optimum value (satisfy the Strong Duality Theorem).

## Another example

Let's take the following problem

$$\begin{array}{lrcrcrcrcr}
\max & 2x_1 & + & 4x_2 & + & 3x_3 & + & x_4 & & \\
\text{subject to} & 3x_1 & + & x_2 & + & x_3 & + & 4x_4 & \leq & 12 \\
 & x_1 & - & 3x_2 & + & 2x_3 & + & 3x_4 & \leq & 7 \\
 & 2x_1 & + & x_2 & + & 3x_3 & - & x_4 & \leq & 10 \\
 & & & & & & & x & \geq & 0
\end{array}$$

The solution to this problem is {42, 0, 10.4, 0, 0.4}. The dual problem is as follows:

$$\begin{array}{lrcrcrcr}
\min & 12y_1 & + & 7y_2 & + & 10y_3 & & \\
\text{subject to} & 3y_1 & + & y_2 & + & 2y_3 & \geq & 2 \\
 & y_1 & - & 3y_2 & + & y_3 & \geq & 4 \\
 & y_1 & + & 2y_2 & + & 3y_3 & \geq & 3 \\
 & 4y_1 & + & 3y_2 & - & y_3 & \geq & 1 \\
 & & & & & y & \geq & 0
\end{array}$$

Step1: feasibility. First constraint is binding, no conclusion for y1. Second constraint is not binding, so y2=0. Third constraint is binding, nothing about y3.

Step2: positive variables. X1=0, nothing to do. X2>0, the second dual constraint binds. X3=0, nothing to do. X4>0, fourth dual constraint binds.

So solution to dual must satisfy: $\begin{cases} y2 = 0 \\ y1 + y3 = 4 \\ 4y1 - y3 = 1 \end{cases}$ with {1, 0, 3} as solution (no invalid constraint and equals optimal value).