# Practice 1.1: Transportation
## *Combinatorial optimization*

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## Special cases : degeneracy

If the basic feasible solution of a transportation problem with m origins and n destinations has fewer than m + n − 1 positive $x_{ij}$ (occupied cells), the problem is said to be a degenerate transportation problem.

Degeneracy can occur at two stages:

1. At the initial solution
2. During the testing of the optimal solution

To resolve degeneracy, we make use of an artificial quantity (*d*). The quantity *d* is assigned to that unoccupied cell, which has the minimum transportation cost.

The use of *d* is illustrated in the following example.

### Example

| Factory | Dealer | | | | Supply |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | |
| A | 2 | 2 | 2 | 4 | 1000 |
| B | 4 | 6 | 4 | 3 | 700 |
| C | 3 | 2 | 1 | 0 | 900 |
| Requirement | 900 | 800 | 500 | 400 | |

### Solution

An initial basic feasible solution is obtained by Matrix Minimum Method.

### Table 1

| Factory | Dealer | | | | Supply |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | |
| A | 2 (900) | 2 (100) | 2 | 4 | 1000 |
| B | 4 | 6 (700) | 4 | 3 | 700 |
| C | 3 | 2 | 1 (500) | 0 (400) | 900 |
| Requirement | 900 | 800 | 500 | 400 | |

Number of basic variables = m + n – 1 = 3 + 4 – 1 = 6
Since number of basic variables is less than 6, therefore, it is a degenerate transportation problem.

To resolve degeneracy, we make use of an artificial quantity(d). The quantity d is assigned to that unoccupied cell, which has the minimum transportation cost.

In the above table, there is a tie in selecting the smallest unoccupied cell. In this situation, you can choose any cell arbitrarily. We select the cell C2 as shown in the following table.

### Table 2

| Factory | Dealer | | | | Supply |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | |
| A | 2 (900) | 2 (100) | 2 | 4 | 1000 |
| B | 4 | 6 (700) | 4 | 3 | 700 |
| C | 3 | 2 (d) | 1 (500) | 0 (400) | 900 + d |
| Requirement | 900 | 800 + d | 500 | 400 | 2600 + d |

Now, we use the stepping stone method to find an optimal solution.

### Calculating opportunity cost

| Unoccupied cells | Increase in cost per unit of reallocation | Remarks |
|---|---|---|
| A3 | +2 – 2 + 2 – 1 = 1 | Cost Increases |
| A4 | +4 – 2 + 2 – 0 = 4 | Cost Increases |
| B1 | +4 – 6 + 2 – 2 = –2 | Cost Decreases |
| B3 | +4 – 6 + 2 – 1 = –1 | Cost Decreases |
| B4 | +3 – 6 + 2 – 0 = –1 | Cost Decreases |
| C1 | +3 – 2 + 2 – 2 = 1 | Cost Increases |

The cell B1 is having the maximum improvement potential, which is equal to -2. The maximum amount that can be allocated to B1 is 700 and this will make the current basic variable corresponding to cell B2 non basic. The improved solution is shown in the following table.

### Table 3

| Factory | Dealer | | | | Supply |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | |
| A | 2 (200) | 2 (800) | 2 | 4 | 1000 |
| B | 4 (700) | 6 | 4 | 3 | 700 |
| C | 3 | 2 (d) | 1 (500) | 0 (400) | 900 |
| Requirement | 900 | 800 | 500 | 400 | 2600 |

The optimal solution is
2 X 200 + 2 X 800 + 4 X 700 + 2 X d + 1 X 500 + 0 X 400 = 5300 + 2d.

Notice that *d* is a very small quantity so it can be neglected in the optimal solution. Thus, the net transportation cost is Rs. 5300

## Special cases: Unbalanced

So far we have assumed that the total supply at the origins is equal to the total requirement at the destinations.
Specifically,

$$\sum_{i=1}^{m} S_i = \sum_{j=1}^{n} D_j$$

But in certain situations, the total supply is not equal to the total demand. Thus, the transportation problem with unequal supply and demand is said to be unbalanced transportation problem.

If the total supply is more than the total demand, we introduce an additional column, which will indicate the surplus supply with transportation cost zero. Similarly, if the total demand is more than the total supply, an additional row is introduced in the table, which represents unsatisfied demand with transportation cost zero. The balancing of an unbalanced transportation problem is illustrated in the following example.

### Example

| Plant | Warehouse | | | Supply |
|---|---|---|---|---|
| | W1 | W2 | W3 | |
| A | 28 | 17 | 26 | 500 |
| B | 19 | 12 | 16 | 300 |
| Demand | 250 | 250 | 500 | |

### Solution:

The total demand is 1000, whereas the total supply is 800.

$$\sum_{i=1}^{m} S_i < \sum_{j=1}^{n} D_j$$

Total supply < total demand.
To solve the problem, we introduce an additional row with transportation cost zero indicating the unsatisfied demand.

| Plant | Warehouse | | | Supply |
|---|---|---|---|---|
| | W1 | W2 | W3 | |
| A | 28 | 17 | 26 | 500 |
| B | 19 | 12 | 16 | 300 |
| Unsatisfied demand | 0 | 0 | 0 | 200 |
| Demand | 250 | 250 | 500 | 1000 |

Using matrix minimum method, we get the following allocations.

| Plant | Warehouse | | | Supply |
|---|---|---|---|---|
| | W1 | W2 | W3 | |
| A | 28 (50) | 17 | 26 (450) | ~~500~~ |
| B | 19 | 12 (250) | 16 (50) | ~~300~~ |
| Unsatisfied demand | 0 (200) | 0 | 0 | ~~200~~ |
| Demand | ~~250~~ | ~~250~~ | ~~500~~ | 1000 |

**Initial basic feasible solution**

50 X 28 + 450 X 26 + 250 X 12 + 50 X 16 + 200 X 0 = 16900.

## Special cases : maximization

There are certain types of transportation problems where the objective function is to be maximized instead of being minimized. These problems can be solved by converting the maximization problem into a minimization problem.

**Example**

Surya Roshni Ltd. has three factories - X, Y, and Z. It supplies goods to four dealers spread all over the country. The production capacities of these factories are 200, 500 and 300 per month respectively.

| Factory | Dealer | | | | Capacity |
|---|---|---|---|---|---|
| | A | B | C | D | |
| X | 12 | 18 | 6 | 25 | 200 |
| Y | 8 | 7 | 10 | 18 | 500 |
| Z | 14 | 3 | 11 | 20 | 300 |
| Demand | 180 | 320 | 100 | 400 | |

Determine a suitable allocation to maximize the total net return.

**Solution.**

Maximization transportation problem can be converted into minimization transportation problem by subtracting each transportation cost from maximum transportation cost.

Here, the maximum transportation cost is 25. So subtract each value from 25. The revised transportation problem is shown below.

**Table 1**

| Factory | Dealer | | | | Capacity |
|---|---|---|---|---|---|
| | A | B | C | D | |
| X | 13 | 7 | 19 | 0 | 200 |
| Y | 17 | 18 | 15 | 7 | 500 |
| Z | 11 | 22 | 14 | 5 | 300 |
| Demand | 180 | 320 | 100 | 400 | |

An initial basic feasible solution is obtained by matrix-minimum method and is shown in the final table.

### Final table

| Factory | Dealer | | | | Capacity |
|---|---|---|---|---|---|
| | A | B | C | D | |
| X | 13 | 7 | 19 | (200) 0 | ~~200~~ |
| Y | (80) 17 | (320) 18 | (100) 15 | 7 | ~~500~~ |
| Z | (100) 11 | 22 | 14 | (200) 5 | ~~300~~ |
| Demand | ~~180~~ | ~~320~~ | ~~100~~ | ~~400~~ | |

### The maximum net return is

25 X 200 + 8 X 80 + 7 X 320 + 10 X 100 + 14 X 100 + 20 X 200 = 14280.

## Special cases : prohibited routes

Sometimes there may be situations, where it is not possible to use certain routes in a transportation problem. For example, road construction, bad road conditions, strike, unexpected floods, local traffic rules, etc. We can handle such type of problems in different ways:

A very large cost represented by M or $\propto$ is assigned to each of such routes, which are not available.
To block the allocation to a cell with a prohibited route, we can cross out that cell.

The problem can then be solved in its usual way

### Example

Consider the following transportation problem.

| Factory | Warehouse | | | Supply |
|---|---|---|---|---|
| | $W_1$ | $W_2$ | $W_3$ | |
| $F_1$ | 16 | $\propto$ | 12 | 200 |
| $F_2$ | 14 | 8 | 18 | 160 |
| $F_3$ | 26 | $\propto$ | 16 | 90 |
| Demand | 180 | 120 | 150 | 450 |

### Solution.

An initial solution is obtained by the matrix minimum method and is shown in the final table.

### Final Table

| Factory | Warehouse | | | Supply |
|---|---|---|---|---|
| | $W_1$ | $W_2$ | $W_3$ | |
| $F_1$ | 16 (50) | ∝ | 12 (150) | ~~200~~ |
| $F_2$ | 14 (40) | 8 (120) | 18 | ~~160~~ |
| $F_3$ | 26 (90) | ∝ | 16 | ~~90~~ |
| Demand | ~~180~~ | ~~120~~ | ~~150~~ | 450 |

### Initial basic feasible solution

16 X 50 + 12 X 150 + 14 X 40 + 8 X 120 + 26 X 90 = 6460.

The minimum transportation cost is Rs. 6460.

## Special cases : time minimization

Succinctly, it is a transportation problem in which the objective is to minimize the time. This problem is same as the transportation problem of minimizing the cost, expect that the unit transportation cost is replaced by the time $t_{ij}$.

### Steps

1. Determine an initial basic feasible solution using any one of the following:

   North West Corner Rule
   Matrix Minimum Method
   Vogel Approximation Method

2. Find $T_k$ for this feasible plan and cross out all the unoccupied cells for which $t_{ij} \geq T_k$.

3. Trace a closed path for the occupied cells corresponding to $T_k$. If no such closed path can be formed, the solution obtained is optimum otherwise, go to step 2.

### Example 1

The following matrix gives data concerning the transportation times $t_{ij}$

| | Destination | | | | | | |
|---|---|---|---|---|---|---|---|
| Origin | D1 | D2 | D3 | D4 | D5 | D6 | Supply |
| O1 | 25 | 30 | 20 | 40 | 45 | 37 | 37 |
| O2 | 30 | 25 | 20 | 30 | 40 | 20 | 22 |
| O3 | 40 | 20 | 40 | 35 | 45 | 22 | 32 |
| O4 | 25 | 24 | 50 | 27 | 30 | 25 | 14 |
| Demand | 15 | 20 | 15 | 25 | 20 | 10 | |

### Solution.

We compute an initial basic feasible solution by north west corner rule which is shown in table 1.

### Table 1

| Destination | | | | | | | |
|---|---|---|---|---|---|---|---|
| **Origin** | **D1** | **D2** | **D3** | **D4** | **D5** | **D6** | **Supply** |
| **O1** | 25 (15) | 30 (20) | 20 (2) | 40 | 45 | 37 | 37 |
| **O2** | 30 | 25 | 20 (13) | 30 (9) | 40 | 20 | 22 |
| **O3** | 40 | 20 | 40 | 35 (16) | 45 (16) | 22 | 32 |
| **O4** | 25 | 24 | 50 | 27 | | | 14 |
| **Demand** | 15 | 20 | 15 | 25 | 20 | 10 | |

Here, $t_{11}$ = 25, $t_{12}$ = 30, $t_{13}$ = 20, $t_{23}$ = 20, $t_{24}$ = 30, $t_{34}$= 35, $t_{35}$ = 45, $t_{45}$ =30, $t_{46}$ = 25

Choose maximum from $t_{ij}$, i.e., $T_1$ = 45. Now, cross out all the unoccupied cells that are ≥ T1.
The unoccupied cell (O3D6) enters into the basis as shown in table 2.

### Table 2

Choose the smallest value with a negative position on the closed path, i.e., 10. Clearly only 10 units can be shifted to the entering cell. The next feasible plan is shown in the following table.

### Table 3

| Destination | | | | | | | |
|---|---|---|---|---|---|---|---|
| **Origin** | **D1** | **D2** | **D3** | **D4** | **D5** | **D6** | **Supply** |
| **O1** | 25 (15) | 30 (20) | 20 (2) | 40 | | 37 | 37 |
| **O2** | 30 | 25 | 20 (13) | 30 (9) | 40 | 20 | 22 |
| **O3** | 40 | 20 | 40 | 35 (16) | | | 32 |
| **O4** | 25 | 24 | | 27 | | 25 | 14 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **Demand** | 15 | 20 | 15 | 25 | 20 | 10 | |

Here, $T_2$ = Max(25, 30, 20, 20, 20, 35, 45, 22, 30) = 45. Now, cross out all the unoccupied cells that are ≥ $T_2$.

### Table 4

By following the same procedure as explained above, we get the following revised matrix.

### Table 6

| | **Destination** | | | | | | |
|---|---|---|---|---|---|---|---|
| **Origin** | **D1** | **D2** | **D3** | **D4** | **D5** | **D6** | **Supply** |
| **O1** | 25 (15) | 30 (20) | 20 (2) | | 45 | 37 | 37 |
| **O2** | 30 | 25 | 20 (13) | | | 20 | 22 |
| **O3** | | 20 | | | 45 | 22 (10) | 32 |
| **O4** | 25 | 24 | 50 | 27 | 30 (14) | 25 | 14 |
| **Demand** | 15 | 20 | 15 | 25 | 20 | 10 | |

$T_3$ = Max(25, 30, 20, 20, 30, 40, 35, 22, 30) = 40. Now, cross out all the unoccupied cells that are ≥ $T_3$.

Now we cannot form any other closed loop with $T_3$.
Hence, the solution obtained at this stage is optimal.
Thus, all the shipments can be made within 40 units.