Tutorial 1: Modelling and basics
Artificial Intelligence

G.Guérard

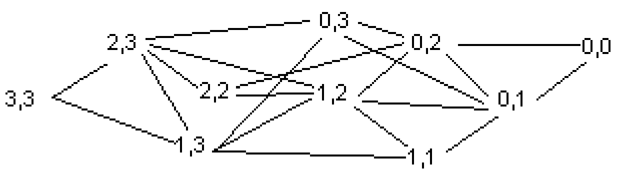
Les étudiants doivent faire des groupes de 3-4 afin de faire un brainstorming pour chaque exercice. Il n'est pas demandé aux étudiants d'avoir une connaissance parfaite de la théorie, vous devez les guider afin qu'ils trouvent une solution aux problèmes. Vous pouvez commencer le TP par de simples questions de cours.

Modelling

Exercise 1

Two players have 2 or more lots of matches. At each turn, the next player may remove a number of matches of a lot (depending on the selected rule). The player who removes the last match loses. Model this game with a graph in the case where one has from two piles each containing three matches, and where a player can remove one or two matches each.
What move the first player must play to win the game?

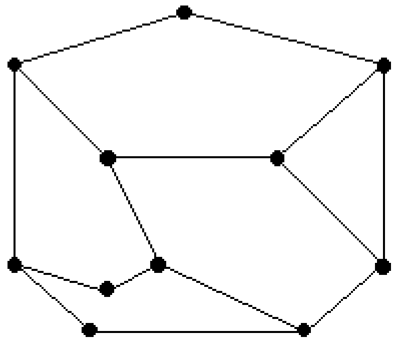
Solution



Edges are oriented from left to right. If we are in position (0,2) or (0,1) we win. Thus, the position (1,3) verify that, whatever play your adversary, you can reach (0,2) or (0,1).

Exercise 2

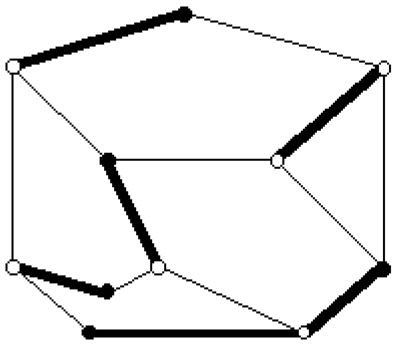
The following graph represents the corridors of a museum. A guard placed in a corridor can monitor two junctions placed at its ends. How many guards are needed (and how to place them) so that all intersections are monitored?
If we now place guards at intersections, assuming that a guard can monitor all corridors leading to this crossroads, how many guards are needed?



Solution

Each guardian will be placed on an edge and can monitor two intersections (vertices). The graph has 11 vertices; it will take at least six guards. We must therefore find a set (minimum) of at least six edges, such that each vertex is incident to at least one of these edges. The graph below provides a solution (thicker edges).

This time, the guards are at the vertices and watch the edges. There must be a set of vertices such that every edge is incident to at least one of these vertices. The graph below provides a solution using 6 vertices (white vertices).



Exercise 3

Is it possible to chart 5 segments on a sheet of paper where each segment cuts exactly 3 others?

Prove that in a non-oriented graph $G = (V, E)$, the sum of all degrees is 2*|E|.

Solution

The vertices of our graph are the segments. An edge between two vertices indicates that the corresponding segments intersect. The degree of each vertex is 3; the sum of the degrees is 15 and should be twice the number of edges.

As 15 is not an even number, it is not possible.

Exercise 4

Skynet is a network with 15 vertices. You can go from every vertex to at least seven other vertices by link. Can we go by link from a vertex X to each of the others?

Solution

Let A be any vertex. X is linked to at least 7 different cities. So we have a network of 8 vertices connected by links. In a vertex, you can also connect to at least seven other vertices; there is therefore a new network of 8 vertices. There must be a shared vertex in these networks, because otherwise the network would have at least 16 vertices. X is connected to A in at most two shots, through this common vertex; it is connected to all other cities.

Exercise 5

A network is composed of seven plants. A city is linked to exactly two plants. Two plants share exactly one city. How many cities are served by the network? Prove that a complete graph with n vertices has $\frac{n(n-1)}{2}$ links.

Solution

Plants are the vertices and cities are represented by an edge connecting two vertices. Rule 2 implies that there are no multiple edge, and that any couple of vertices are adjacent; i.e. that the graph is complete; it's K_7, i.e. 21 links.

Exercise 6

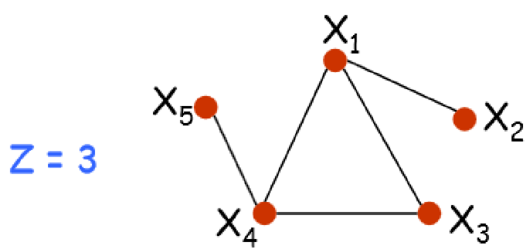
Considering the following mathematical system:

$$\begin{cases} X_1 + X_2 \leq 1 \\ X_1 + X_3 + X_4 \leq 1 \\ X_4 + X_5 \leq 1 \end{cases}$$

Where X_i is 0 or 1. Determine, with a graph problem, a solution that maximizes the objective function: $Z = X_1 + X_2 + X_3 + X_4 + X_5$.

Solution

Vertex represents a variable. Two vertices are connected if they are in the same inequality. The problem is to find a maximum set of vertices where no couple of vertices has a direct link.

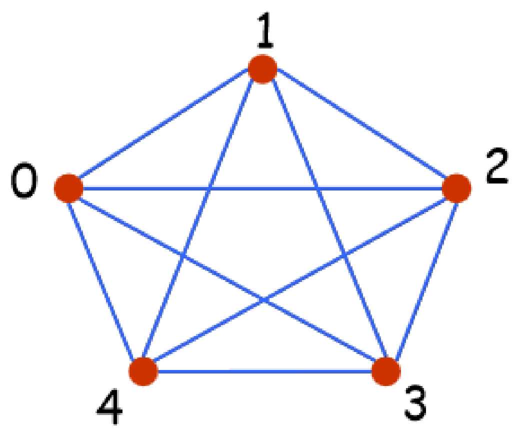


Exercise 7

Considering a dominoes game using the numbers 0, 1, 2, 3, 4, as on each domino include two distinct digits, such as 1 and 3, the following problem is proposed: Is it possible to align all the dominoes so that when two pawns "touch" the numbers "in contact" are identical?

Solution

Represent the problem as a complete graph K5. The problem is to find an Eulerian cycle in the graph.



0 1	1 2	2 3	3 4	4 0	0 2	2 4	4 1	1 3	3 0

Exercise 8

A group of 9 students met every day at a round table. How many days can they come together if we want anyone not having twice the same neighbor?

Solution

Denote 1, 2… 9 the people and consider the complete graph K_9. A composition of the table corresponds to a Hamiltonian cycle in K_9 (passing once through each vertex). If two compositions have a common edge, this means that the two connected people by this edge are sat side by side. Thus, the problem is to determine the number of disjoint Hamiltonian cycles in K_9.

The K9 graph with 9 x 8/2 = 36 edges and each cycle using edges 9, this number is at most equal to 4. It is actually 4, as shown by the following 4 cycles disjoint Hamiltonian: 1,2,3,9,4,8,5,7,6 — 1,3,4,2,5,9,6,8,7 — 1,4,5,3,6,1,7,9,8 — 1,5,6,4,7,2,8,1,9

Tree

Exercise 9

Let $G = (S, A)$ be a non-oriented graph with n vertices. Prove that all the following properties are equivalent:

- G is a tree
- G is connected and if we remove an edge, G is no longer connected
- G is connected with $n - 1$ edge
- G has no cycle until we add an edge
- G has no cycle with $n - 1$ edge
- Only one path between any couple of vertices

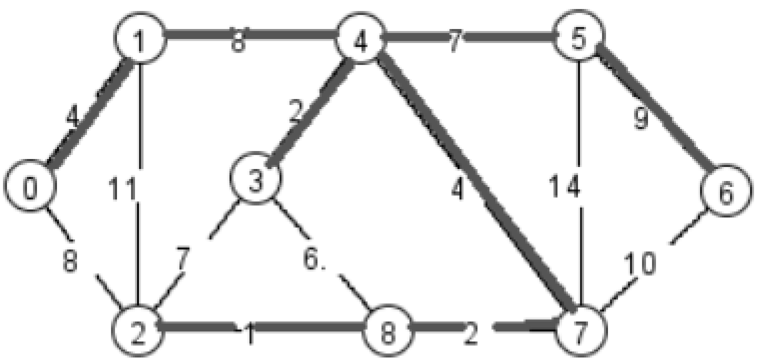
Exercise 10

To increase efficiency of local plants, a company needs to link them in a connected network. The cost of the line between a couple of plants is shown in this adjacency matrix (lines are non-directed):

-	4	8	-	-	-	-	-	-
	-	11	-	8	-	-	-	-
		-	7	-	-	-	-	1
			-	2	-	-	-	6
				-	7	-	4	-
					-	9	14	-
						-	10	-
							-	2
								-

Construct an associated graph. Propose an algorithm to build a tree. How to solve the problem at the minimum cost? Give a solution. Found a solution with the maximum cost.

Solution

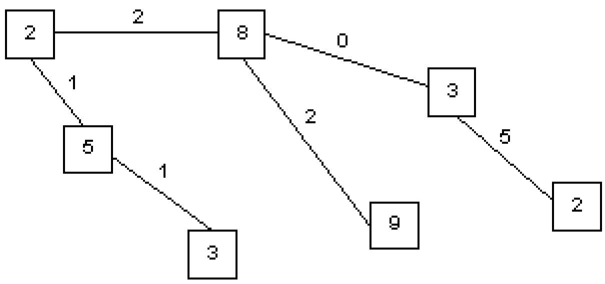


Prim or Kruskal

For the maximum cost, use 1/c for each cost c and find an MST.

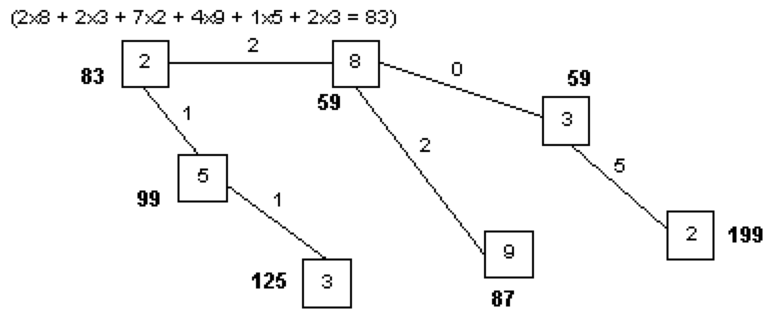
Exercise 11

We want to add a battery in a network. The following graph shows cost to send energy between two substations in the network, and the amount of energy sent by a substation. You have to place the battery at a substation while minimizing total cost.



Correction

Calculate the sum of all cost at each substation, i.e. the cost of each path between any substation to the chosen substation (edge*vertex).



Graph coloring

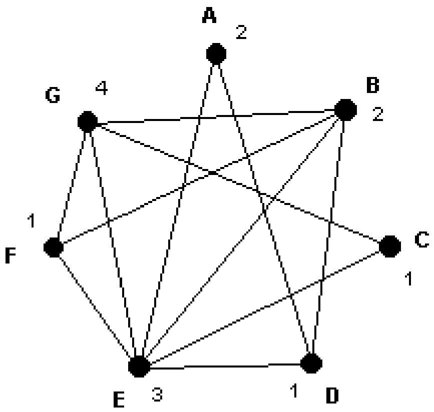
Exercise 12

A server can route a maximum of x packages at the same time. Seven substations are linked to a server, a substation cannot send packages if some substations already use the link. The next table presents each substation of the ability to send a package in function of the other ones. For example, a package from A cannot be sent if there is already a package from D but can be sent when B sent a package.

Substation	A	B	C	D	E	F	G
Is not allowed with	D,E	D,E,F,G	E,G	A,B,E	A,B,C,D,F,G	B,E,G	B,C,E,F

Represent the links in a graph. How many packages the server has to manage at the same time (maximum value)?

Solution



The graph is 4-chromatic.

Exercise 13

A school must pass written tests to four students: Pierre, Jean, Guillermo and Ibrahim. Seven disciplines are involved: mathematics, physics, biology, French, English, Spanish and history. Pierre must pass the mathematics, physics and English, Jean mathematics, biology and French, Guillermo mathematics, English and Spanish and Ibrahim physics, French and history. What is the minimum number of time slots to be expected for no student has to pass two tests simultaneously? What is the chromatic number of a complete graph? How to bound chromatic number in a graph?

Solution

A vertex represents a discipline, an edge links two disciplines if they cannot take place at the same time. The problem can be solved by searching a minimum number of colors. The largest complete sub graph is K3, so, there are at least 3 colors. The maximum degree is five, so there are at most 5 colors. By experimentation, the graph is 3-chromatic.