

Tutorial 1: Simplex method

Combinatorial Optimization

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Vous pouvez commencer le TP par de simples questions de cours ou des exemples « sans réflexion ».

Primal-Dual

Exercise 1

The advertising alternative for a company include television, radio, and newspaper advertisements. The costs and estimates for audience coverage are given in the table below.

	Television	Radio	Newspaper
Cost per advertisement	£2000	£300	£600
Audience per advertisement	100000	18000	40000

The local newspaper limits the number of weekly advertisements from a single company to ten. Moreover, in order to balance the advertising among the three types of media, no more than half of the total number of advertisements should occur on the radio, and at least 10% should occur on television. The weekly advertising budget is £18'200. How many advertisements should be run in each of the three types of media to maximize the total audience?

Solution

Let x_1 , x_2 , x_3 represent the number of advertisements in television, newspaper and radio respectively. The objective function to maximize is therefore $z=100'000x_1+40'000x_2+18'000x_3$ under constraints:

$$20x_1 + 6x_2 + 3x_3 \leq 182$$

$$x_2 \leq 10$$

$$-x_1 - x_2 + x_3 \leq 0$$

$$-9x_1 + x_2 + x_3 \leq 0$$

Thus, the initial simplex table is as follows

x_1	x_2	x_3	s_1	s_2	s_3	s_4	b
20	6	3	1	0	0	0	182
0	1	0	0	1	0	0	10
-1	-1	1	0	0	1	0	0
-9	1	1	0	0	0	1	0
-100,000	-40,000	-18,000	0	0	0	0	0

An optimal solution is as follows:

Media	Number of Advertisements	Cost	Audience
Television	4	\$ 8,000	400,000
Newspaper	10	\$ 6,000	400,000
Radio	14	\$ 4,200	252,000
Total	28	\$18,200	1,052,000

Exercise 2

A small petroleum company owns two refineries. Refinery 1 costs £20'000 per day to operate, and it can produce 400 barrels of high-grade oil, 300 barrels of medium-grade oil, and 200 barrels of low-grade oil each day. Refinery 2 costs £25'000 per day to operate, and it can produce 300, 400, 500 barrels (same order) each day. The company has orders totaling 25'000 barrels of high-grade oil, 27'000 barrels of medium-grade oil, and 30'000 barrels of low-grade oil. How many days should it run each refinery to minimize its costs and still refine enough oil to meet its orders?

Solution

Let x_1 and x_2 represent the number of days the two refineries are operated. Then the total cost is given by $C=20'000x_1+25'000x_2$ under constraints:

$$\text{(High-grade)} \quad 400x_1 + 300x_2 \geq 25,000$$

$$\text{(Medium-grade)} \quad 300x_1 + 400x_2 \geq 27,000$$

$$\text{(Low-grade)} \quad 200x_1 + 500x_2 \geq 30,000$$

The dual simplex table is as follows:

y_1	y_2	y_3	s_1	s_2	b
400	300	200	1	0	20,000
300	400	500	0	1	25,000
-25,000	-27,000	-30,000	0	0	0

That gives the following results:

y_1	y_2	y_3	s_1	s_2	b	Basic Variables
1	$\frac{1}{2}$	0	$\frac{1}{280}$	$-\frac{1}{700}$	$\frac{250}{7}$	y_1
0	$\frac{1}{2}$	1	$-\frac{3}{1400}$	$\frac{1}{350}$	$\frac{200}{7}$	y_3
0	500	0	25	50	1,750,000	

By CS, we find that $x_1=25$ days and $x_2=50$ days. Produced barrels are as follows: {25'000, 27'500, 30'000}. As CS indicates, the second inequality is not reach; we have an extra 500 barrels.

Exercise 3

Consider the following LP problem:

$$\begin{aligned}
 \min \quad & -5x_1 - 4x_2 - 3x_3 \\
 \text{s.t.} \quad & 2x_1 + 3x_2 + 1x_3 \geq 5 \\
 & 4x_1 + 1x_2 + 2x_3 \leq 13 \\
 & 3x_1 + 4x_2 + 2x_3 = 9 \\
 & x_1, x_2, x_3 \geq 0
 \end{aligned}$$

Use the CS conditions to verify whether or not the solution {1,1,1} is a, optimal solution to the LP problem.

Solution

The dual is:

$$\begin{aligned}
 \max \quad & 5p_1 + 13p_2 + 9p_3 \\
 \text{s.t.} \quad & 2p_1 + 4p_2 + 3p_3 \leq -5 \\
 & 3p_1 + 1p_2 + 4p_3 \leq -4 \\
 & 1p_1 + 2p_2 + 2p_3 \leq -3 \\
 & p_1 \geq 0, p_2 \leq 0
 \end{aligned}$$

We first not the {1,1,1} in primal is feasible, and constraints 1 and 2 are not binding. The CS conditions require that $p_1=p_2=0$. Substituting $p_1=p_2=0$ obtain an inconsistent system. This indicates that the complementary slackness conditions are not satisfied by {1,1,1}, wherefore it is not an optimal solution.

Exercise 4

Find the optimal value z of the LP problem below by inspecting its dual. Then, find the primal optimal solution using the CS conditions.

$$\begin{aligned} \min z &= 10x_1 + 4x_2 + 5x_3 \\ \text{s.t.} \quad &5x_1 - 7x_2 + 3x_3 \geq 50 \\ &x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution

The dual of the problem is:

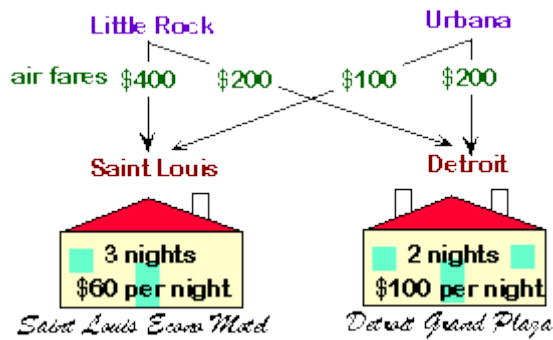
$$\begin{aligned} \max y &= 50p \\ \text{s.t.} \quad &5p \leq 10 \\ &-7p \leq 4 \\ &3p \leq 5 \\ &p \geq 0 \end{aligned}$$

The first two dual constraints are redundant and can be removed (the third and the fourth one are more restrictive). Since the objective value increases when p increases and we are maximizing, the optimal dual solution is $p^*=5/3$ with $y=250/3$.

As we have seen, the first and second constraint are redundant, thus there are not binding at p^* , wherefore the corresponding primal variables must be zero, that is $x_1=x_2=0$. Substituting these values to the primal constraint we get $x_3=50/3$. We have found a feasible solution $\{0,0,50/3\}$ which satisfies the CS conditions and $z=250/3=y$, by strong duality, this is an optimal solution.

Exercise 5

Your software company has launched the latest version of its web browser, "Java Cruise 4.0." As sales manager, you are planning to promote Java Cruise 4.0 by sending sales forces to software conventions running concurrently in Saint Louis and Detroit. You have 6 representatives available at each of your Little Rock, Ark. and Urbana, Ill. branches, and you would like to send at least 5 to the Saint Louis convention and at least 4 to the Detroit convention. The Saint Louis convention will last for three days, while the Detroit convention will last for two days. Air fares (per person) and hotel accommodation costs (per person) are shown in the following figure.



How many representatives should you send from each branch to each convention in order to minimize the total (air travel and accommodation) cost? What will the total cost amount to?

Solution

Let x = the number of representatives sent from Little Rock to St. Louis, y = the number of representatives sent from Little Rock to Detroit, z = the number of representatives sent from Urbana to St. Louis, w = the number of representatives sent from Urbana to Detroit.

Minimize $C = 580x + 400y + 280z + 400w$ subject to

$$x + y \leq 6 ; z + w \leq 6 ; x + z \geq 5 ; y + w \geq 4 ; x \geq 0, y \geq 0, z \geq 0, w \geq 0$$

An optimal solution is $x = 0, y = 4, z = 5, w = 0; C = 3,000$

Thus, you should send 4 representatives from Little Rock to Detroit and 5 from Urbana to St. Louis, for a total cost of \$3,000.

Shadow costs

Exercise 6

Consider the following Linear Programming model:

$$\begin{aligned} \text{MAX} \quad & X + 2Y \\ \text{S.A.} \quad & 2X + 4Y \leq 12 \\ & 4X + 3Y \leq 16 \\ & X \geq 0 \quad Y \geq 0 \end{aligned}$$

Find a solution that maximizes also the value X .

Solution

The initial tableau with S_1 and S_2 as initial basic variables looks like this:

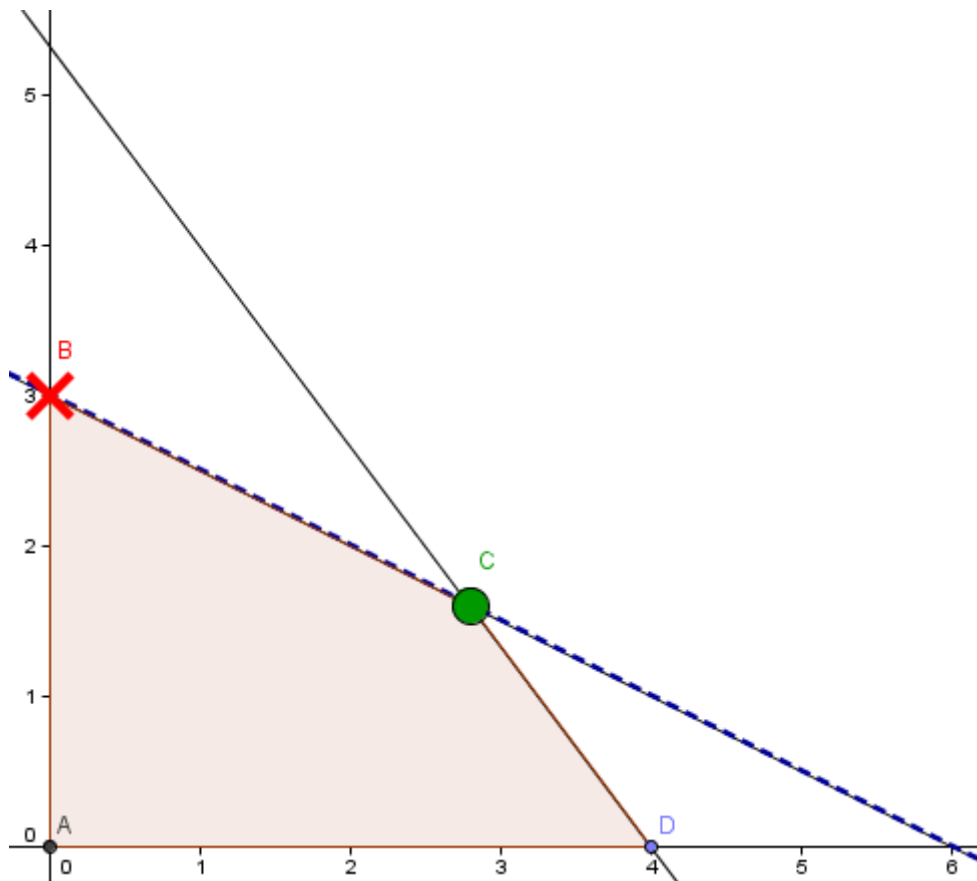
X	Y	S1	S2	
2	4	1	0	12
4	3	0	1	16
-1	-2	0	0	0

Y enters in the base. For determining the variable that leaves the base we use the minimum quotient cut: $\text{Min } \{12/4; 16/3\}=3 \implies \text{S1 leaves the base}$. With this information we update the table:

X	Y	S1	S2	
1/2	1	1/4	0	3
5/2	0	-3/4	1	7
0	0	1/2	0	6

After one iteration of the Simplex Method we find the optimal solution, where Y and S2 are basic variables. The optimal solution is $X=0, Y=3, S1=0, S2=7$. The optimal value is $V(P)=6$. Note that X (a non-basic variable) has **zero reduced cost** that determines the existence of **multiple or infinite optimal solutions**, so the current solution is one of the optimum vertex.

The diagram below shows the graphic resolution of the problem where the optimal solution we have encountered in applying the Simplex Method corresponds to vertex B. Note that the dotted blue line corresponds to the contour of the objective function with the same slope as constraint 1 (slope $-1/2$).



4. Assume we increase the objective coefficient of variable x_1 by 0.5. Is the current basis still optimal? What happens to the optimal value?
5. Assume we increase the objective coefficient of variable x_4 by 3. Is the current basis still optimal? What happens to the optimal value?
6. Assume we increase the objective coefficient of variable x_3 by 3. Is the current basis still optimal? What happens to the optimal value?

Solution

1. Looking at the sensitivity analysis for constraint 1, we see that an increase of 0.5 is within the allowable range of change (+2/3 is OK). Thus, the basis will not change. Now, recall the equation for the objective value: $z = y^T b$. Thus, we only have to look at the corresponding dual variables in the sensitivity analysis to compute the change of the objective value. The total cost changes by $0.5 \cdot (-10) = -5$. Thus, the new cost is 7.
 2. This change is also in the allowable range of change. Thus, looking at the dual variables we know that the total cost changes by $-1 \cdot 7 = -7$. Thus, the new cost is 5.
 3. Looking at the sensitivity analysis, we see that an increase of 1 is outside of the allowable range and the optimal basis will change. Thus, we cannot answer this question directly with the information present.
1. By looking at the sensitivity analysis for variable x_1 we see that an increase of 0.5 is within the allowable range of change (x_1 can be between 1.5 and 5.6666667). Thus, the current basis remains optimal. The new optimal value changes by $0.5 \cdot 2 = 1$. Thus the new total value is 13.
 2. The cost increase is within the allowable range for variable x_4 . Thus, the current basis remains optimal. Because x_4 is a non-basic variable, i.e. it is set equal to 0, this change in the cost coefficient does not change the objective value.
 3. This cost increase is outside of the allowable range for variable x_3 (var.up was -10). Thus, the optimal basis will change and we cannot answer this question fully without doing more computation.