Artificial Intelligence

Combinatorial Optimization

G. Guérard

Department of Nouvelles Energies
Ecole Supérieur d'Ingénieurs Léonard de Vinci

Lecture 1

Outline

Linear Programming

Motivation

Linear programming is a tool for **optimal allocation** of scarce **resources among** a number of competing **activities**.

Applications

Shortest path, network flow, MST, matching, assignment, games, etc. It is a widely applicable problem-solving model with fast solvers available ans applications in all areas (finance, marketing, physics, logistics, energy, computer science...).

Example

A brewery produces ale and beer. The production is limited by scarce resources: corn, hops and malt. Recipes for ale and beer require different proportions of those resources.

	corn(kg)	hops(oz)	malt(kgs)	profit(£)
available	480	160	1190	
ale(for 1 barrel)	5	4	35	13
beer (idem)	15	4	20	23

PROBLEM: *choose product mix to mazimise profits.*

Example

Some examples of allocations, the number in red is the BOTTLENECK.

	corn	hops	malt	profit
only ale (34)	179	136	1190	420
only beer (32)	480	128	1100	720
mix (12 / 28)	480	160	980	800
?				>800?

Example

The mathematical formulation of the problem is as follows: *let A be the number of barrels of beer and B be the number of barrels of ale.*

$$\begin{cases} max & 13A + 23B \ \ (profit) \\ s.t & 5A + 15B \leq 480 \ \ (corn) \\ & 4A + 4B \leq 160 \ \ (hops) \\ & 35A + 20B \leq 1190 \ \ (malt) \\ & A \geq 0, \ B \geq 0 \end{cases}$$

Feasible region

Feasible region

A FEASIBLE REGION, FEASIBLE SET, SEARCH SPACE, or SOLUTION SPACE is the **set of all possible points** (sets of values of the choice variables) of an optimization problem that satisfy the problem's constraints, potentially including inequalities, equalities, and integer constraints. This is the initial set of candidate solutions to the problem.

Convex set

In linear programming problems, the feasible set is a **convex polytope**: *a region in multidimensional space whose boundaries are formed by hyperplanes and whose corners are vertices*. A convex feasible set is one in which a line segment connecting any two feasible points goes through only other feasible points, and not through any points outside the feasible set.

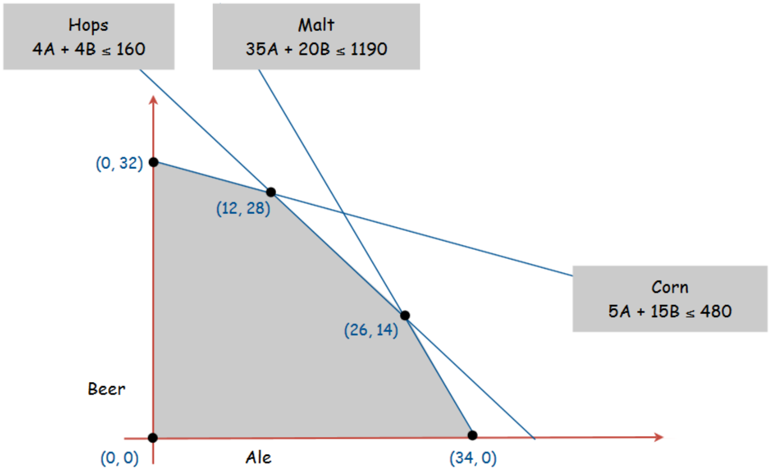
Feasible region

1 - The first step is to **sketch** the lines that correspond to the boundaries of the regions. These are found by replacing the $\leq$ and $\geq$ inequalities by an equal sign, $=$. Do not place the objective function on the graph. We are graphing only the constraints or linear inequalities on this graph.

2 - The second step is to **determine the feasible region** by shading. Shading an inequality results in shading a half-plane. For most of the inequalities that we have from linear programming problems, it's pretty easy to tell which direction to shade just by looking at the inequality. Take a point not in the tested boundary. If this point verifies the inequation, then shade its side. The origin $(0, 0)$ is usually an easy point to use as long as it's not a solution to the linear equation.

Feasible region

Feasible set of the brewery problem:



Optimal solution

Theorem

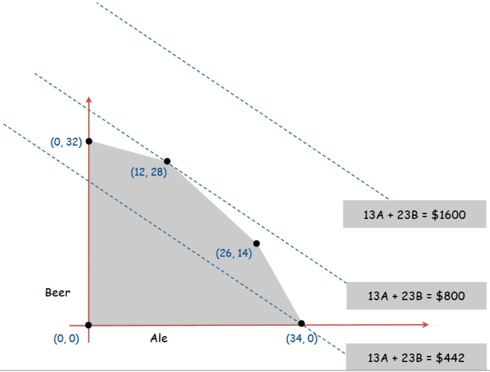
Due to optimal conditions and a convex feasible region, regardless of objective function coefficents, **an optimal solution occurs at an extreme point**.

In other words, the solution is going to be on the edge of the region, not in the middle. An intersection is a vertex of the feasible region (unique point of linear system between two or more equation).

By this theorem, how to solve the maximization problem?

Optimal solution

If you sketch a line corresponding to the objective function coefficient anywhere, by taking any point on this line you find its value. Any point of the objective function which is also a point in the feasible region have this value. The problem is to find an extreme point wich maximize the value of the objective function. We have to know how evolve its value in the feasible region.



Optimal solution

Normal vector

A NORMAL is an object such as a line or vector that is **perpendicular to a given object**. For example, in the two-dimensional case, the normal line to a curve at a given point is the line perpendicular to the tangent line to the curve at the point.

Linear hyperplane

If the surface is given by a Cartesian equation $f(x, y, z) = 0$, with a class $C1$ function f, a point on the surface is regular if the gradient f is zero at this point. It is then the gradient vector itself is a normal vector: $\overrightarrow{grad}\, f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}$.

The value of the hyperplane increases in the direction of the normal vector.

Standard form linear program

INPUT: real numbers a_{ij}, c_j, b_i.
OUTPUT: real numbers x_j.
A problem has n nonnegative variables and m constraints. It is a maximization of linear objective function subject to linear equations.

$$\begin{cases} \textit{maximize} & c_1x_1 + \ldots + c_nx_n \\ s.t. & a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ & \vdots \\ & a_{m1}x_1 + \ldots + a_{mn}x_n = b_m \\ & x_1, \ldots, x_n \geq 0 \end{cases} \iff \begin{cases} \textit{Maximize} & c^T x \\ s.t. & Ax = b \\ & x \geq 0 \\ \textit{matrix version} & \end{cases}$$

Standard form linear program

When you have a constraint as: $a_1x_1 + \ldots + a_nx_n \leq b$, add a new dimension to tranform the inequality to an equality: $a_1x_1 + \ldots + a_nx_n + x_{n+1} = b$.
Same with: $a_1x_1 + \ldots + a_nx_n \geq b$ but first multiply all by -1: $-a_1x_1 - \ldots - a_nx_n + x_{n+1} = -b$. Then, add variable Z and equation corresponding to objective function.

maximize	13A	+	23B		
subject to the constraints	5A	+	15B	≤	480
	4A	+	4B	≤	160
	35A	+	20B	≤	1190
			A, B	≥	0

become

maximize	Z											
subject to the constraints	13A	+	23B							− Z	=	0
	5A	+	15B	+	S_C						=	480
	4A	+	4B			+	S_H				=	160
	35A	+	20B					+	S_M		=	1190
			A, B, S_C, S_H, S_M								≥	0

Extreme point property

As cited previsouly, an optimal solution is an extreme point. The set of extreme points of the feasible region is exactly the set of all basic feasible solutions of the linear programming problem. Once a hyperplan can cut every hyperplan defined by each constraint, *the number of extreme points is exponential*.

Greedy property

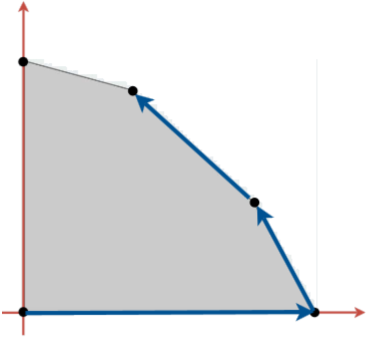
An EXTREME POINT is optimal iff no neighboring extreme point is better.

FIND AN ALGORITHM THAT VISITS NEIGHBORING EXTREME POINT IN ORDER TO INCREASE OBJECTIVE FUNCTION. IF NO NEIGHBORING EXTREME POINT IS BETTER, THEN WE HAVE FOUND AN OPTIMAL POINT.

Simplex algorithm

George DANTZIG developed the greatest and most successful algorithm of all time, *solving linear program quickly and determine perturbation's results* as the same time.

1. *Start at some extreme point.*
2. *Pivot from one extreme point to a neighboring one.*
3. *Repeat until optimal.*

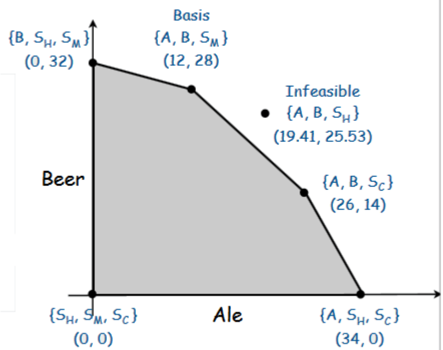


Basis

There are two ways to find a basis (subset of m of the n variables).

Basic Feasible Solution

Set $n - m$ nonbasic variables to 0, solve for remaining m variables. Solve m equations in m unknowns. If it give a unique solution, it is an extreme point.

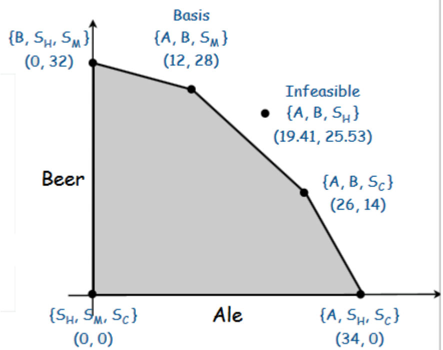


Basis

There are two ways to find a basis (subset of m of the n variables).

Initial Basic Feasible Solution

Set non-basis variables to zero ($A = 0$, $B = 0$, $Z = 0$). The equations give directly the basis variables' value. The extreme point on simplex is the origin.



Pivot

We choose B as pivot $B = (1/15)(480 - 5A - S_c)$. Rewrite 2nd equation and eliminate B in first, third and fourth equations. We put B into the basis, S_C is out ($S_C = 0$).

maximize Z

subject to the constraints

$$
\begin{aligned}
(16/3)A & & - (23/15)\,S_C & & - Z &= -736 \\
(1/3)\,A + B & & + (1/15)\,S_C & & &= 32 \\
(8/3)\,A & & - (4/15)\,S_C + S_H & & &= 32 \\
(85/3)\,A & & - (4/3)\,S_C & + S_M & &= 550 \\
A,\ B,\ S_C,\ S_H,\ S_M & & & & &\geq 0
\end{aligned}
$$

Why B? *Best coefficient in objective function.*
Why in first equation? *To preverse the feasibility by choosing the minimum ratio:* $min\{480/15,\ 160/4,\ 1190/20\}$.

Pivot

We choose A as pivot $B = (3/8)(32 + 4/5 * S_c - S_H)$. Rewrite 3rd equation and eliminate A in first, second and fourth equations. We put A into the basis, S_H is out ($S_H = 0$).

maximize	Z										
subject to the constraints			−	S_C	−	$2S_H$			− Z	=	-800
		B	+	$(1/10)\ S_C$	+	$(1/8)\ S_H$				=	28
	A		−	$(1/10)\ S_C$	+	$(3/8)\ S_H$				=	12
			−	$(25/6)\ S_C$	−	$(85/8)\ S_H$	+	S_M		=	110
				A, B, S_C, S_H, S_M						≥	0

Optimality

When to stop Pivoting? *When all coefficients in top row are non-positive.*
How to know which variable is out? *The one which appears in the Z-equation.*
Why is resulting optimal solution? *Any feasible solution satisfies system of equations, in particular the Z-equation. Here the Z-equation is:* $Z = 800 - S_c - 2S_H$. *Since* $S_C, S_H \geq 0$, *and we seek maximum value for* Z, *the optimal objective value* $Z^* = 800$ *with both non-basis variables equal to zero.*

Simplex table

In order to use the Dantzig's idea in any maximization problem ($max\ f = -min - f$), we use a table. Here, the objective function is write like: $Z - 13A - 23B = 0$.

Base	A	B	S_C	S_H	S_M	b	Cb
S_C	5	15	1	0	0	480	
S_H	4	4	0	1	0	160	
S_M	35	20	0	0	1	1190	
Z	-13	-23	0	0	0	0	

Where b is the non-variable part of the equations.

Basis

We choose the column where the coefficient in the Z-equation is the lower possible (negative). Here we choose B. Then, Cb is equal to, for each row, the coefficient of the non-variable b divided by the variable part B. We choose the lowest non-negative row.

Base	A	B	S_C	S_H	S_M	b	Cb
S_C	5	15	1	0	0	480	480/15
S_H	4	4	0	1	0	160	160/4
S_M	35	20	0	0	1	1190	1190/20
Z	-13	-23	0	0	0	0	

Pivot

Then, we divide the row of the pivot by its coefficient, and the other coefficient of the column are updated by zero. Other values are updated derictly from the previous table by the cross-multiplication:
$new\ value = previous\ value - \frac{row\ projection * column\ projection}{pivot}$. Let's take an example with the second row:

Previous raw	4	4	0	1	0	160
	-	-	-	-	-	-
projection	5*4	4*15	1*4	0*4	0*4	480*4
	/	/	/	/	/	/
pivot	15	15	15	15	15	15
	=	=	=	=	=	=
result	8/3	0	-4/15	1	0	32

Pivot

There is the table after computing the pivot formula. S_C left the basis instead of B.

Base	A	B	S_C	S_H	S_M	b	Cb
B	1/3	1	1/15	0	0	32	
S_H	8/3	0	-4/15	1	0	32	
S_M	85/3	0	-4/3	0	1	550	
Z	-16/3	0	23/15	0	0	736	

Optimality

The algorithm continues while it exists a negative value in the Z-equation.

Base	A	B	S_C	S_H	S_M	b	Cb
B	1/3	1	1/15	0	0	32	32*3
S_H	8/3	0	-4/15	1	0	32	32*3/8
S_M	85/3	0	-4/3	0	1	550	550*3/85
Z	-16/3	0	23/15	0	0	736	

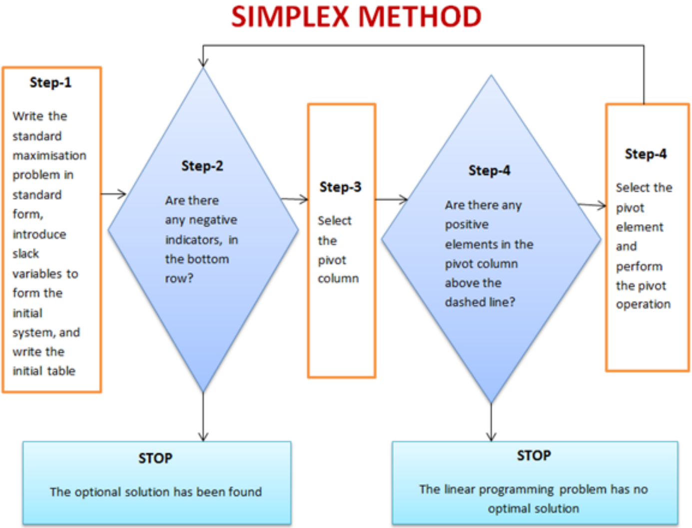
End

Base	A	B	S_C	S_H	S_M	b	Cb
B	0	1	1/10	1/8	0	28	
A	1	0	-1/10	3/8	0	12	
S_M	0	0	-25/6	-85/8	1	110	
Z	0	0	1	2	0	800	

CONCLUSION:

$\{Z = 800,\ A = 12,\ B = 28,\ S_C = 0,\ S_H = 0,\ S_M = 110\}$

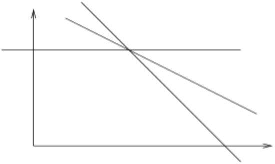
Overview


SIMPLEX METHOD
Step-1
Write the standard maximisation problem in standard form, introduce slack variables to form the initial system, and write the initial table
Step-2
Are there any negative indicators, in the bottom row?
Step-3
Select the pivot column
Step-4
Are there any positive elements in the pivot column above the dashed line?
Step-4
Select the pivot element and perform the pivot operation
STOP
The optional solution has been found
STOP
The linear programming problem has no optimal solution

Degeneracy

If the smallest *Cb* qotient is zero, the value of a basic variable will become zero in the next iteration. The objective value will not improve in this iteration, and we can cycling around bases. The reason of the DEGENERACY is a *redundant constraint on the feasible set*. The solution is degenerate.

$$\begin{array}{rcrcl} x_1 & + & x_2 & \leq & 6 \\ & & x_2 & \leq & 3 \\ x_1 & + & 2x_2 & \leq & 9 \\ & & x_1,\ x_2 & \geq & 0 \end{array}$$



Degeneracy

A method to obtain a optimal solution to the degenerate problem is to compute the DUAL program. We will see later the dual program is also used after primal program to guarantee the optimality. The dual program transform the primal program as follows (*dual of dual is primal*):

Primal :

$$\max \sum_{j=1}^{n} c_j x_j$$

$$\text{s.à.} \sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = 1, ..., m$$

$$x_j \geq 0, j = 1, ..., n$$

Dual:

$$\min \sum_{j=1}^{n} b_i y_i$$

$$\text{s.à.} \sum_{i=m}^{n} a_{ij} y_i \geq c_j, j = 1, ..., n$$

$$y_i \geq 0, i = 1, ..., m$$

Primal :

$$\max_{x \in \mathbb{R}^n} c^T x$$

$$\text{s.à.} \quad Ax \leq b$$

$$x \geq 0$$

Dual:

$$\min \quad b^T y$$

$$\text{s.à.} \quad A^T y \geq c$$

$$y \geq 0$$

Primal into dual form

maximization	$\Leftrightarrow$	minimization
dual program	$\Leftrightarrow$	primal program
primal program	$\Leftrightarrow$	dual program
constraints		*variables*
$\geq$	$\Leftrightarrow$	≤ 0
$\leq$	$\Leftrightarrow$	≥ 0
$=$	$\Leftrightarrow$	free
variables		*constraints*
≥ 0	$\Leftrightarrow$	$\geq$
≤ 0	$\Leftrightarrow$	$\leq$
free	$\Leftrightarrow$	$=$

Duality properties

Weak duality

Let x be a feasible point in the primal (minimization) and y be a feasible point in the dual (maximization). Then $c^T x \geq b^T y$.

Strong duality

In a pair of primal and dual linear programs, if one of them has an optimal solution, so does the other, and their optimal values are equal: $c^T x = b^T y$.

Strong duality

HOW TO FIND AN OPTIMAL SOLUTION IN A PROGRAM FROM THE DUAL (OR PRIMAL) OPTIMAL SOLUTION?

Complementary slackness (CS)

Assume primal has a solution x^* and dual has a solution y^*.

1. *If $x_j^* > 0$, then the j-th constraint in dual is binding.*
2. *If the j-th constraint in dual is not binding, then $x_j^* = 0$.*
3. *If $y_i^* > 0$, then the i-th constraint in primal is binding.*
4. *If the i-th constraint in primal is not binding, then $y_i^* = 0$.*

Fundamental knowledge

YOU HAVE TO KNOW BEFORE THE TUTORIAL:

1. *Geometric resolution:*
   1. *Feasible domain and extreme point definitions.*
   2. *How to find an optimal solution.*
   3. *Standard form linear program.*
2. *Simplex algorithm:*
   1. *Basic feasible solution.*
   2. *Pivot*
   3. *Optimality.*
3. *Dual form:*
   1. *How to form the dual simplex.*
   2. *Weak duality and strong duality.*
   3. *Complementary slackness.*